

A STRONG CONSTANT RANK THEOREM FOR SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we establish a strong constant rank theorem for convex solutions of the semilinear elliptic equation $\Delta u = G(u)$. For every $2 \leq k \leq n$, under the conditions $G > 0$, $G' < 0$ and $GG'' \leq k(G')^2/(k-1)$, we prove that if $\sigma_k(\nabla^2 u)$ attains a local minimum, then it vanishes identically and $\nabla^2 u$ has constant rank at most $k-1$. This resolves the question left open by Li and Xu [8] for the intermediate cases $3 \leq k \leq n-1$. The main step in our proof is to handle the third derivative terms using the concavity of suitable Hessian operators and the Lagrange multiplier method.

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1. INTRODUCTION

The constant rank theorem is an important tool in geometric partial differential equations and asserts that the Hessian of a convex solution has constant rank in the domain under suitable structural conditions. It was first developed in the study of convexity of solutions to semilinear elliptic equations by Caffarelli and Friedman [4] and Korevaar and Lewis [7]. Further developments for fully nonlinear elliptic equations include the work of Guan and Ma [6] in the study of the Christoffel-Minkowski problem, as well as that of Caffarelli, Guan and Ma [3], Bian and Guan [2] and others.

Constant rank theorems have also been used to classify entire concave solutions of the Liouville equation

$$(1.1) \quad -\Delta u = e^{2u} \quad \text{in } \mathbb{R}^n.$$

Bergweiler, Eremenko and Langley [1] proved that every such solution in dimension two is one-dimensional. Using the constant rank theorem, Eremenko, Gui, Li and Xu [5] obtained the same conclusion in arbitrary dimensions under the additional assumption that the solution attains its supremum. Li and Xu [8, Question 1.1] asked whether this assumption can be removed. Motivated by this question, they studied convex solutions of

$$(1.2) \quad \Delta u = G(u)$$

with $G > 0$, $G' < 0$ and $GG'' \leq A(G')^2$. They proved that if $\sigma_2(\nabla^2 u)$ attains a local minimum, then $\nabla^2 u$ has rank one when $A \leq 2$, while the corresponding condition for

$\sigma_n(\nabla^2 u)$ implies that it vanishes identically when $A \leq n/(n-1)$. Here, σ_k denotes the k -th elementary symmetric function of the eigenvalues of the Hessian; see Section 2 for its definition. The corresponding question for the intermediate cases $3 \leq k \leq n-1$ was left open.

In this paper, we resolve the remaining cases and establish a unified result for every $2 \leq k \leq n$ under the natural condition $A \leq k/(k-1)$. Our main theorem is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in C^4(\Omega)$ be a convex solution of (1.2) in Ω . Suppose that G is a C^2 function and satisfies*

$$(1.3) \quad G > 0, \quad G' < 0, \quad GG'' \leq A(G')^2, \quad 0 < A \leq \frac{k}{k-1},$$

where $2 \leq k \leq n$. If $\sigma_k(\nabla^2 u)$ attains a local minimum at some point $x_0 \in \Omega$, then

$$\sigma_k(\nabla^2 u) \equiv 0 \quad \text{in } \Omega.$$

Moreover, $\nabla^2 u$ has constant rank r in Ω , where $1 \leq r \leq k-1$.

This result strengthens the classical constant rank theorem [7] by replacing the assumption that $\sigma_k(\nabla^2 u)$ vanishes at some point with the weaker assumption that it attains a local minimum.

The proof is motivated by our earlier studies of constant rank theorems for special Lagrangian type equations [11, 12], where we used the Lagrange multiplier method to estimate the third derivative terms under the constraints obtained by differentiating the equation. We use the same method here. In the present problem, at a local minimum of $\sigma_k(\nabla^2 u)$, the identity $\nabla \sigma_k(\nabla^2 u) = 0$ provides an additional linear constraint. Combining these constraints with the concavity of the suitable Hessian operators, we apply the Lagrange multiplier method to estimate the third derivative terms, which leads to the structural condition (1.3).

As an application, Theorem 1.1 provides a sufficient condition for an affirmative answer to Question 1.1 in [8]. Indeed, if u is an entire concave solution of (1.1), then $-u$ satisfies $\Delta(-u) = G(-u)$, where $G(t) = e^{-2t}$ and $GG'' = (G')^2$. Theorem 1.1, together with the result of Korevaar and Lewis [7, Theorem 2], implies that u depends on at most two variables whenever $\sigma_3(-\nabla^2 u)$ attains a local minimum. The two-dimensional classification of Bergweiler, Eremenko and Langley [1] then yields the following result.

Corollary 1.2. *Let $n \geq 3$, and let $u \in C^4(\mathbb{R}^n)$ be a concave solution of (1.1). If $\sigma_3(-\nabla^2 u)$ attains a local minimum, then u is one-dimensional.*

2. PRELIMINARIES

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and $1 \leq k \leq n$, the k -th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}.$$

We also adopt the convention that $\sigma_0(\lambda) = 1$ and $\sigma_{-1}(\lambda) = \sigma_{n+1}(\lambda) = 0$. The definition of σ_k for vectors in $\mathbb{R}^n$ can be extended to $n \times n$ symmetric matrices in a natural way by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \lambda_2(W), \dots, \lambda_n(W))$ are the eigenvalues of the symmetric matrix W . We denote by $\sigma_{k-1}(\lambda|i)$ the $(k-1)$ -th elementary symmetric function obtained by setting $\lambda_i = 0$, and by $\sigma_{k-2}(\lambda|ij)$ ($i \neq j$) the $(k-2)$ -th elementary symmetric function obtained by setting $\lambda_i = \lambda_j = 0$. When W is diagonal, we use the same notation $\sigma_{k-1}(W|i)$ and $\sigma_{k-2}(W|ij)$ for the corresponding symmetric functions of its diagonal entries. The first and second derivatives of the elementary symmetric functions are given in the following lemma.

Lemma 2.1 (Proposition 2.2 in [6]). *Suppose that $W = (w_{\alpha\beta})$ is diagonal. For $k = 1, 2, \dots, n$, we have*

$$(2.1) \quad \frac{\partial \sigma_k(W)}{\partial w_{\alpha\beta}} = \begin{cases} \sigma_{k-1}(W|\alpha) & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

and

$$\frac{\partial^2 \sigma_k(W)}{\partial w_{\alpha\beta} \partial w_{\gamma\delta}} = \begin{cases} \sigma_{k-2}(W|\alpha\gamma) & \text{if } \alpha = \beta, \gamma = \delta, \alpha \neq \gamma, \\ -\sigma_{k-2}(W|\alpha\gamma) & \text{if } \alpha = \delta, \beta = \gamma, \alpha \neq \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For $W = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and a symmetric matrix $B = (B_{ij})$, it follows from Lemma 2.1 and the chain rule that

$$(2.2) \quad \begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \sigma_k(W + tB) &= \sum_i \sigma_{k-1}(\lambda|i) B_{ii}, \\ \left. \frac{d^2}{dt^2} \right|_{t=0} \sigma_k(W + tB) &= \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) (B_{ii} B_{jj} - B_{ij}^2). \end{aligned}$$

We also use the elementary identities below; see [9, Lemma 2.14] or [10, Section 2.5].

$$(2.3) \quad \begin{aligned} \sigma_k(\lambda) &= \lambda_i \sigma_{k-1}(\lambda|i) + \sigma_k(\lambda|i), \\ \sum_i \lambda_i \sigma_{k-1}(\lambda|i) &= k \sigma_k(\lambda), \\ \sigma_k(\lambda|i) - \sigma_k(\lambda|j) &= (\lambda_j - \lambda_i) \sigma_{k-1}(\lambda|ij), \quad i \neq j. \end{aligned}$$

For $1 \leq k \leq n$, define

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for } 1 \leq j \leq k\}.$$

These are open convex cones satisfying

$$\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1, \quad \Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } i\}.$$

Finally, we state the classical concavity properties of elementary symmetric functions, which play a key role in our proof. We refer the reader to [9] for a proof.

Lemma 2.2. *For $1 \leq k \leq n$, the functions $\sigma_k^{1/k}(\lambda)$ and $\sigma_k(\lambda)/\sigma_{k-1}(\lambda)$ are concave in Γ_k .*

3. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.1. The initial calculations follow Section 4 of Li and Xu [8] and are included for completeness. The new part of the proof combines the concavity properties in Lemma 2.2 with the Lagrange multiplier method to estimate the third derivative terms for every $2 \leq k \leq n$.

Proof of Theorem 1.1. Define the auxiliary function

$$\phi(x) = \sigma_k(\nabla^2 u(x)),$$

and let $x_0 \in \Omega$ be a local minimum point of ϕ . Since $A \leq k/(k-1) \leq 2$, the classical constant rank theorem of Korevaar and Lewis [7, Theorem 1] implies that $\nabla^2 u$ has constant rank in Ω . Hence it suffices to prove that $\phi(x_0) = 0$. Suppose to the contrary that $\phi(x_0) > 0$. By choosing orthonormal coordinates, we may assume that $\nabla^2 u(x_0)$ is diagonal, with eigenvalues $\lambda_i = u_{ii}(x_0)$, $i = 1, 2, \dots, n$. The following calculations will be carried out at a fixed point x_0 .

Taking the first derivatives of $\phi(x)$ and applying (2.1), we obtain at the local minimum point x_0

$$(3.1) \quad 0 = \phi_\alpha = \sum_{i,j} \frac{\partial \sigma_k}{\partial u_{ij}} u_{ij\alpha} = \sum_i \sigma_{k-1}(\lambda|i) u_{ii\alpha} \quad \text{for } \alpha = 1, 2, \dots, n.$$

Differentiating once more and applying Lemma 2.1, we have

$$\begin{aligned} \phi_{\alpha\alpha} &= \sum_{i,j} \frac{\partial \sigma_k}{\partial u_{ij}} u_{ij\alpha\alpha} + \sum_{i,j,p,q} \frac{\partial^2 \sigma_k}{\partial u_{ij} \partial u_{pq}} u_{ij\alpha} u_{pq\alpha} \\ &= \sum_i \sigma_{k-1}(\lambda|i) u_{ii\alpha\alpha} + \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) (u_{ii\alpha} u_{jj\alpha} - u_{ij\alpha}^2). \end{aligned}$$

Consequently,

$$(3.2) \quad \Delta \phi = \sum_i \sigma_{k-1}(\lambda|i) \Delta u_{ii} + \sum_\alpha \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) (u_{ii\alpha} u_{jj\alpha} - u_{ij\alpha}^2).$$

To handle the fourth-order derivative terms in (3.2), we differentiate equation (1.2) with respect to x_α and obtain

$$(3.3) \quad \sum_i u_{ii\alpha} = G' u_\alpha$$

and

$$(3.4) \quad \sum_i u_{ii\alpha} = G'' u_\alpha^2 + G' u_{\alpha\alpha}.$$

Substituting (3.4) into (3.2) and using the second identity of (2.3), we conclude that

$$\Delta\phi - kG'\phi = G'' \sum_i \sigma_{k-1}(\lambda|i) u_i^2 + \sum_\alpha \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) (u_{ii\alpha} u_{jj\alpha} - u_{ij\alpha}^2).$$

Moreover,

$$(3.5) \quad \begin{aligned} \Delta\phi - kG'\phi &\leq G'' \sum_i \sigma_{k-1}(\lambda|i) u_i^2 \\ &+ \sum_\alpha \left(\sum_{i \neq j} \sigma_{k-2}(\lambda|ij) u_{ii\alpha} u_{jj\alpha} - 2 \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) u_{ii\alpha}^2 \right). \end{aligned}$$

We next estimate the third-order derivative terms in (3.5) under the constraints (3.1), assuming first that all the λ_i are positive. Fix $\alpha \in \{1, 2, \dots, n\}$, and set $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ with

$$\xi_i = u_{ii\alpha}, \quad i = 1, 2, \dots, n.$$

Then (3.1) takes the form

$$(3.6) \quad \sum_i \sigma_{k-1}(\lambda|i) \xi_i = 0.$$

To estimate the term $\sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j$, we consider the function

$$(3.7) \quad \lambda \mapsto \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)},$$

which is concave in Γ_n . Indeed, by the first identity in (2.3),

$$\frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)} = \lambda_\alpha + \frac{\sigma_k(\lambda|\alpha)}{\sigma_{k-1}(\lambda|\alpha)}.$$

The claim now follows from Lemma 2.2. Differentiating the quotient (3.7) twice along $\lambda + t\xi$ and using the constraint (3.6) together with (2.2), we obtain

$$\begin{aligned} 0 &\geq \frac{d^2}{dt^2} \Big|_{t=0} \frac{\sigma_k(\lambda + t\xi)}{\sigma_{k-1}(\lambda + t\xi|\alpha)} \\ &= \frac{1}{\sigma_{k-1}(\lambda|\alpha)} \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j - \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)^2} \sum_{\substack{i, j \neq \alpha \\ i \neq j}} \sigma_{k-3}(\lambda|ij\alpha) \xi_i \xi_j \\ &\quad + \frac{2\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)^3} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2. \end{aligned}$$

Consequently,

$$(3.8) \quad \begin{aligned} \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j &\leq \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)} \sum_{\substack{i,j \neq \alpha \\ i \neq j}} \sigma_{k-3}(\lambda|ij\alpha) \xi_i \xi_j \\ &\quad - \frac{2\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)^2} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2. \end{aligned}$$

On the other hand, by Lemma 2.2, the concavity of $\sigma_{k-1}(\lambda|\alpha)^{1/(k-1)}$ implies that

$$\begin{aligned} 0 &\geq \frac{d^2}{dt^2} \Big|_{t=0} \sigma_{k-1}(\lambda + t\xi|\alpha)^{1/(k-1)} \\ &= \frac{\sigma_{k-1}(\lambda|\alpha)^{1/(k-1)-1}}{k-1} \left(\sum_{\substack{i,j \neq \alpha \\ i \neq j}} \sigma_{k-3}(\lambda|ij\alpha) \xi_i \xi_j \right. \\ &\quad \left. - \frac{k-2}{k-1} \frac{1}{\sigma_{k-1}(\lambda|\alpha)} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2 \right). \end{aligned}$$

Thus

$$(3.9) \quad \sum_{\substack{i,j \neq \alpha \\ i \neq j}} \sigma_{k-3}(\lambda|ij\alpha) \xi_i \xi_j \leq \frac{k-2}{k-1} \frac{1}{\sigma_{k-1}(\lambda|\alpha)} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2.$$

Combining (3.8) and (3.9), we find

$$\begin{aligned} \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j &\leq -\frac{k}{k-1} \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda|\alpha)^2} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2 \\ &\leq -\frac{k}{k-1} \frac{\lambda_\alpha}{\sigma_{k-1}(\lambda|\alpha)} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2, \end{aligned}$$

where we used the first identity in (2.3). Since $k/(k-1) \leq 2$, it follows that

$$(3.10) \quad \begin{aligned} &\sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j - 2 \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i^2 \\ &\leq -\frac{k}{k-1} \left(\frac{\lambda_\alpha}{\sigma_{k-1}(\lambda|\alpha)} \left(\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i \right)^2 + \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i^2 \right). \end{aligned}$$

We now estimate the right-hand side of (3.10) by the Lagrange multiplier method under the constraints (3.3) and (3.6). Set

$$X_i = \xi_i, \quad i \neq \alpha, \quad X_\alpha = -\frac{\sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i}{\sigma_{k-1}(\lambda|\alpha)},$$

and define the quadratic polynomial

$$f(X_1, X_2, \dots, X_n) = - \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) X_i^2 - \lambda_\alpha \sigma_{k-1}(\lambda|\alpha) X_\alpha^2.$$

Moreover, it follows from (3.3), (3.6) and the last identity in (2.3) that

$$\begin{aligned} \sigma_{k-1}(\lambda|\alpha) G' u_\alpha &= \sigma_{k-1}(\lambda|\alpha) \sum_i \xi_i = \sum_i (\sigma_{k-1}(\lambda|\alpha) - \sigma_{k-1}(\lambda|i)) \xi_i \\ &= \sum_{i \neq \alpha} (\lambda_i - \lambda_\alpha) \sigma_{k-2}(\lambda|i\alpha) \xi_i \\ &= \sum_{i \neq \alpha} \lambda_i \sigma_{k-2}(\lambda|i\alpha) X_i + \lambda_\alpha \sigma_{k-1}(\lambda|\alpha) X_\alpha. \end{aligned}$$

It therefore suffices to find the maximum of $f(X_1, X_2, \dots, X_n)$ subject to the constraint

$$(3.11) \quad g(X_1, X_2, \dots, X_n) = 0,$$

where

$$g(X_1, X_2, \dots, X_n) = \sum_{i \neq \alpha} \lambda_i \sigma_{k-2}(\lambda|i\alpha) X_i + \lambda_\alpha \sigma_{k-1}(\lambda|\alpha) X_\alpha - \sigma_{k-1}(\lambda|\alpha) G' u_\alpha.$$

For $s \in \mathbb{R}$, define the associated Lagrangian L by

$$L(X_1, X_2, \dots, X_n, s) = f(X_1, X_2, \dots, X_n) + 2sg(X_1, X_2, \dots, X_n).$$

Taking partial derivatives of L , we obtain

$$\begin{aligned} \frac{\partial L}{\partial X_i} &= -2\sigma_{k-2}(\lambda|i\alpha) X_i + 2s\lambda_i \sigma_{k-2}(\lambda|i\alpha), \quad i \neq \alpha, \\ \frac{\partial L}{\partial X_\alpha} &= -2\lambda_\alpha \sigma_{k-1}(\lambda|\alpha) X_\alpha + 2s\lambda_\alpha \sigma_{k-1}(\lambda|\alpha), \end{aligned}$$

and

$$\frac{\partial L}{\partial s} = 2g(X_1, X_2, \dots, X_n).$$

The vanishing of these derivatives implies that the stationary point satisfies

$$(3.12) \quad X_i = s\lambda_i, \quad i \neq \alpha, \quad X_\alpha = s.$$

The constraint $g = 0$ then determines s as

$$s = \frac{\sigma_{k-1}(\lambda|\alpha) G' u_\alpha}{\lambda_\alpha \sigma_{k-1}(\lambda|\alpha) + \sum_{i \neq \alpha} \lambda_i^2 \sigma_{k-2}(\lambda|i\alpha)}.$$

We next verify that the stationary point (3.12) maximizes f under (3.11). For every nonzero vector $\eta = (\eta_1, \eta_2, \dots, \eta_n)$,

$$\sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j} \eta_i \eta_j = -2 \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \eta_i^2 - 2\lambda_\alpha \sigma_{k-1}(\lambda|\alpha) \eta_\alpha^2 < 0,$$

so the Hessian of f is negative definite. Since g is affine, (3.12) is the unique global maximizer under the constraint (3.11). Evaluating f at this point, we find

$$(3.13) \quad \max_{\{g=0\}} f = -\frac{\sigma_{k-1}(\lambda|\alpha)^2 (G')^2 u_\alpha^2}{\lambda_\alpha \sigma_{k-1}(\lambda|\alpha) + \sum_{i \neq \alpha} \lambda_i^2 \sigma_{k-2}(\lambda|i\alpha)}.$$

By the first two identities in (2.3),

$$\begin{aligned} \sum_{i \neq \alpha} \lambda_i^2 \sigma_{k-2}(\lambda|i\alpha) &= \sum_{i \neq \alpha} \lambda_i (\sigma_{k-1}(\lambda|\alpha) - \sigma_{k-1}(\lambda|i\alpha)) \\ &= (\sigma_1(\lambda) - \lambda_\alpha) \sigma_{k-1}(\lambda|\alpha) - k \sigma_k(\lambda|\alpha). \end{aligned}$$

Consequently,

$$\lambda_\alpha \sigma_{k-1}(\lambda|\alpha) + \sum_{i \neq \alpha} \lambda_i^2 \sigma_{k-2}(\lambda|i\alpha) \leq \sigma_1(\lambda) \sigma_{k-1}(\lambda|\alpha).$$

It follows from (3.13) that

$$f(X_1, X_2, \dots, X_n) \leq -\frac{\sigma_{k-1}(\lambda|\alpha)}{\sigma_1(\lambda)} (G')^2 u_\alpha^2$$

subject to (3.11). Returning to (3.10), we obtain

$$(3.14) \quad \sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j - 2 \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i^2 \leq -\frac{k}{k-1} \frac{\sigma_{k-1}(\lambda|\alpha)}{\sigma_1(\lambda)} (G')^2 u_\alpha^2.$$

To handle zero eigenvalues, recall that $\sigma_k(\lambda) > 0$, so $\sigma_{k-1}(\lambda|\alpha) > 0$ for every α . Let

$$\lambda^\varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \dots, \lambda_n + \varepsilon), \quad \varepsilon > 0,$$

keep $\xi_i^\varepsilon = \xi_i$ for $i \neq \alpha$, and define

$$\xi_\alpha^\varepsilon = -\frac{\sum_{i \neq \alpha} \sigma_{k-1}(\lambda^\varepsilon|i) \xi_i}{\sigma_{k-1}(\lambda^\varepsilon|\alpha)}.$$

Then $\sum_i \sigma_{k-1}(\lambda^\varepsilon|i) \xi_i^\varepsilon = 0$ and $\xi^\varepsilon \rightarrow \xi$ as $\varepsilon \rightarrow 0$. Applying (3.14) to $(\lambda^\varepsilon, \xi^\varepsilon)$, with $G' u_\alpha$ replaced by $\sum_i \xi_i^\varepsilon$, and letting $\varepsilon \rightarrow 0$, we obtain the desired estimate by (3.3).

Finally, recalling that $\xi_i = u_{ii\alpha}$ and $\sigma_1(\lambda) = G$, we have

$$\sum_{i \neq j} \sigma_{k-2}(\lambda|ij) \xi_i \xi_j - 2 \sum_{i \neq \alpha} \sigma_{k-2}(\lambda|i\alpha) \xi_i^2 \leq -\frac{k}{k-1} \frac{(G')^2}{G} \sigma_{k-1}(\lambda|\alpha) u_\alpha^2.$$

Together with (3.5), this estimate implies that

$$\Delta\phi - kG'\phi \leq \left(G'' - \frac{k}{k-1} \frac{(G')^2}{G} \right) \sum_{\alpha} \sigma_{k-1}(\lambda|\alpha) u_\alpha^2.$$

By the structural condition (1.3), we conclude that $\Delta\phi \leq kG'\phi$. Since $G' < 0$ and $\phi(x_0) > 0$, this yields

$$0 \leq \Delta\phi(x_0) \leq kG'(u(x_0))\phi(x_0) < 0,$$

a contradiction. Hence $\phi(x_0) = 0$.

Since $\nabla^2 u$ has constant rank r in Ω , the identity $\sigma_k(\nabla^2 u(x_0)) = 0$ and $\nabla^2 u \geq 0$ imply that $r \leq k - 1$. Moreover, $\Delta u = G > 0$ gives $r \geq 1$. It follows that $\sigma_k(\nabla^2 u) \equiv 0$ in Ω , which completes the proof. $\square$

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