

CLASSIFICATION OF ENTIRE CONCAVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we classify entire concave solutions of the semilinear equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n.$$

Assuming $f > 0$, $f' \geq cf$ and $ff'' \leq A(f')^2$ for some constants $c > 0$ and $0 < A < 2$, we prove that every such solution is one-dimensional. This includes the classical Liouville equation as a special case. We first prove the result when a concave solution attains a local maximum, then use this result together with concavity to find a monotone direction, establish stability and derive a contradiction.

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1. INTRODUCTION

The classification of entire solutions to semilinear elliptic equations is a fundamental problem. A classical example is the De Giorgi conjecture, which asks whether every bounded solution of the Allen-Cahn equation $-\Delta u = u - u^3$ in $\mathbb{R}^n$ that is strictly monotone in one direction must be one-dimensional at least when $n \leq 8$. This conjecture was proved in dimensions two and three by Ghoussoub and Gui [7] and Ambrosio and Cabré [1], respectively. More recently, independent proofs of the conjecture in dimension four were given by Liu, Luo, Wang, J. Wei, Y. Wei and Wu [11] and by Chan, Fernández-Real, Figalli, Florit-Simon and Serra [3]. In higher dimensions, Savin [12] proved the conclusion up to dimension eight under an additional limiting assumption, while del Pino, Kowalczyk and Wei [4] constructed counterexamples in dimensions nine and higher.

In this paper, we consider entire concave solutions of

$$(1.1) \quad -\Delta u = f(u) \quad \text{in } \mathbb{R}^n.$$

Our aim is to find structural conditions on f under which every such solution is one-dimensional in the sense that $u(x) = g(x \cdot e)$ for some unit vector $e \in \mathbb{R}^n$ and some function g on $\mathbb{R}$. A classical example of equation (1.1) is the Liouville equation

$$(1.2) \quad -\Delta u = e^{2u} \quad \text{in } \mathbb{R}^n.$$

In dimension two, this equation describes conformal metrics of constant Gaussian curvature 1, while in higher dimensions it arises in the study of the gravitational equilibrium of polytropic stars, as discussed by Farina [6].

As shown in the proof of Corollary 1.2 below, every one-dimensional solution of (1.2) has the form (1.4) and reduces to $u(x) = -\log \cosh(x \cdot e)$ after translation and scaling. Its derivative in the direction e is $-\tanh(x \cdot e)$, which solves the Allen-Cahn equation after rescaling the independent variable. In view of this connection, the monotonicity condition in the De Giorgi conjecture suggests imposing a corresponding sign condition on $\nabla^2 u$ for (1.2). Since $\Delta u = -e^{2u} < 0$, concavity is the natural choice, motivating the study of entire concave solutions of (1.2). In dimension two, Bergweiler, Eremenko and Langley [2] proved that every such solution is one-dimensional. Using the constant rank theorem of Korevaar and Lewis [9], Eremenko, Gui, Li and Xu [5] obtained the same conclusion in arbitrary dimensions under the additional assumption that the solution attains a local maximum. Li and Xu [10, Question 1.1] asked whether this assumption can be removed. The following theorem answers this question in a more general setting.

Theorem 1.1. *Let $n \geq 1$ and $f \in C^\infty(\mathbb{R})$ satisfy*

$$(1.3) \quad f > 0, \quad f' \geq cf, \quad ff'' \leq A(f')^2 \quad \text{on } \mathbb{R},$$

for some constants $c > 0$ and $0 < A < 2$. Then every concave solution $u \in C^4(\mathbb{R}^n)$ of (1.1) is one-dimensional.

We briefly describe the proof strategy. In Section 2, we first establish a generalization of Theorem 1.10 of Eremenko, Gui, Li and Xu [5] for equation (1.1), proving that every entire concave solution attaining a local maximum is one-dimensional. The proof relies on the constant rank theorem of Korevaar and Lewis [9], which motivates the structural condition $ff'' \leq A(f')^2$, with $0 < A < 2$, in (1.3). To complete the proof of our main result in Section 3, we argue by contradiction and choose the smallest dimension $n \geq 2$ admitting an entire concave solution u that is not one-dimensional. Such a solution cannot attain a local maximum, and concavity allows us to find a direction in which u is strictly increasing and hence establish its stability. Motivated by the work of Ambrosio and Cabré [1], we then consider the translates $u_\ell(x) = u(x', x_n + \ell)$, where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $\ell > 0$. Using stability, concavity and the structural condition $f' \geq cf$, with $c > 0$, we obtain local bounds for u_ℓ independent of ℓ , which together with its monotonicity in ℓ ensure that u_ℓ converges pointwise to a finite limit U . Standard elliptic regularity theory then shows that U is a concave solution of (1.1), and local convergence preserves its stability. Moreover, U is independent of x_n and remains stable when regarded as a solution in $\mathbb{R}^{n-1}$. On the other hand, the choice of n implies that U is one-dimensional, contradicting the fact that every one-dimensional solution of (1.1) is unstable.

Applying Theorem 1.1 to $f(s) = e^{2s}$, we obtain the following explicit classification of the entire concave solutions of the Liouville equation (1.2), answering Question 1.1 of Li and Xu [10].

Corollary 1.2. *Every concave solution $u \in C^4(\mathbb{R}^n)$ of (1.2) has the form*

$$(1.4) \quad u(x) = \log a - \log \cosh(a(x \cdot e - b)),$$

where $a > 0$, $b \in \mathbb{R}$ and $e \in \mathbb{R}^n$ with $|e| = 1$.

2. CLASSIFICATION OF CONCAVE SOLUTIONS WITH A LOCAL MAXIMUM

In this section, we extend Theorem 1.10 of Eremenko, Gui, Li and Xu [5] to equation (1.1), proving that, under the assumptions on f in Theorem 1.1, every entire concave solution attaining a local maximum is one-dimensional.

To describe the one-dimensional solutions of (1.1), we consider the initial value problem

$$(2.1) \quad -g'' = f(g), \quad g(0) = m, \quad g'(0) = 0,$$

for each $m \in \mathbb{R}$. The assumptions $f > 0$ and $f' \geq cf$ imply that f is strictly increasing and that

$$(2.2) \quad f(s) \leq f(m)e^{c(s-m)} \quad \text{for } s \leq m,$$

while

$$f(s) \geq f(m)e^{c(s-m)} \quad \text{for } s \geq m.$$

These bounds show that f is a bijection from $\mathbb{R}$ onto $(0, \infty)$, with

$$(2.3) \quad \int_{-\infty}^m f(s)ds \leq \frac{f(m)}{c} < \infty.$$

Lemma 2.1. *For each $m \in \mathbb{R}$, the solution of (2.1) exists on $\mathbb{R}$, and every entire solution of $-g'' = f(g)$ is a translate of one of these solutions.*

Proof. Fix $m \in \mathbb{R}$, and let g be the local solution of (2.1). Since $g'' = -f(g) < 0$ and $g'(0) = 0$, we have $g \leq m$. Multiplying the equation by $2g'$ and integrating from 0 to s , we obtain

$$(g'(s))^2 = 2 \int_{g(s)}^m f(\tau)d\tau \leq \frac{2f(m)}{c},$$

where the inequality follows from (2.3). Consequently,

$$|g'(s)| \leq \sqrt{2f(m)/c}, \quad \text{and } |g(s) - m| \leq |s|\sqrt{2f(m)/c}.$$

By standard continuation theory for ordinary differential equations, the estimates above allow the solution g to be extended past any finite endpoint of its maximal existence interval, and hence g is entire. Since g is strictly concave and $g'(0) = 0$, the point 0 is the unique maximum point, while uniqueness for the initial value problem shows that $g(-s) = g(s)$.

To prove the second assertion, let g be an arbitrary entire solution of $-g'' = f(g)$. It remains to show that g' vanishes somewhere. Since $g'' < 0$, g' is strictly decreasing and therefore either has a unique zero or has a fixed sign. If $g' > 0$ everywhere, then $g(\sigma) \geq g(0)$ for $\sigma \geq 0$, and $f' > 0$ yields

$$g'(s) = g'(0) - \int_0^s f(g(\sigma)) d\sigma \leq g'(0) - sf(g(0)) < 0$$

for sufficiently large s , a contradiction. The case $g' < 0$ everywhere leads to the same contradiction by integrating toward $-\infty$. Thus g' vanishes at a unique point b . Taking $m = g(b)$, the translated function $g(s + b)$ solves (2.1), so g is a translate of its unique solution. $\square$

It follows from Lemma 2.1 that every entire one-dimensional solution of (1.1) attains a global maximum and has the form

$$(2.4) \quad u(x) = g(x \cdot e - b),$$

where $|e| = 1$, $b \in \mathbb{R}$ and g solves (2.1) with $m = \max_{\mathbb{R}^n} u$. We now turn to the main result of this section.

Lemma 2.2. *If a concave solution $u \in C^4(\mathbb{R}^n)$ of (1.1) attains a local maximum, then u is one-dimensional.*

Proof. The conclusion is immediate when $n = 1$, so assume $n \geq 2$. After a translation, suppose that a local maximum is attained at 0. Then $\nabla u(0) = 0$, and concavity implies that $u(0) = \max_{\mathbb{R}^n} u$. Observe that the convex function $-u$ solves $\Delta(-u) = f(u)$, where $f(-s)$ is positive, strictly decreasing and satisfies

$$f(-s)f''(-s) \leq A(f'(-s))^2 < 2(f'(-s))^2.$$

It follows from the constant rank theorem of Korevaar and Lewis [9, Theorem 1] that $\nabla^2 u$ has constant rank. Their Theorem 2 further shows that the directions in which $\nabla^2 u$ vanishes do not depend on x .

Writing $r = \text{rank} \nabla^2 u$, we have $r \geq 1$ since $-\Delta u = f(u) > 0$. Choosing orthonormal coordinates so that $\nabla^2 u$ vanishes in the last $n - r$ coordinate directions at every point, we obtain

$$u_{ij} = 0 \quad \text{for } i = r + 1, r + 2, \dots, n, \quad j = 1, 2, \dots, n.$$

Thus $\nabla u_i = 0$ and each u_i with $i > r$ is constant. As $u_i(0) = 0$, these derivatives vanish identically, and u depends only on the first r variables. Moreover, we can regard u as a function on $\mathbb{R}^r$, where it satisfies

$$-\Delta u = f(u), \quad \nabla^2 u < 0, \quad \text{and } u(0) = \max_{\mathbb{R}^r} u.$$

Suppose that $r \geq 2$. Since u is strictly concave and attains its maximum at 0, there exists $\varepsilon > 0$ such that

$$\max_{|x|=1} u(x) \leq u(0) - \varepsilon.$$

For $|x| \geq 1$, use the identity

$$\frac{x}{|x|} = \left(1 - \frac{1}{|x|}\right) 0 + \frac{1}{|x|} x$$

in the concavity inequality to obtain

$$u(0) - \varepsilon \geq u\left(\frac{x}{|x|}\right) \geq \left(1 - \frac{1}{|x|}\right) u(0) + \frac{u(x)}{|x|},$$

and hence

$$(2.5) \quad u(x) \leq u(0) - \varepsilon|x|.$$

Applying (2.2) with $m = u(0)$, we find $f(u(x)) \leq f(u(0))e^{-c\varepsilon|x|}$ outside B_1 . Therefore, by polar integration,

$$(2.6) \quad \int_{\mathbb{R}^r} f(u) dx \leq |B_1| f(u(0)) + f(u(0)) |\partial B_1| \int_1^\infty e^{-c\varepsilon\rho} \rho^{r-1} d\rho < \infty.$$

We next estimate the same integral from below. By concavity, $u(0) \leq u(x) - \nabla u(x) \cdot x$. For $x \in \partial B_R$, where $R \geq 1$, the outer unit normal is $\nu = x/R$, and (2.5) yields

$$\frac{\partial u}{\partial \nu}(x) = \frac{\nabla u(x) \cdot x}{R} \leq \frac{u(x) - u(0)}{R} \leq -\varepsilon.$$

Integrating equation (1.1) and applying the divergence theorem, we conclude that

$$\begin{aligned} \int_{B_R} f(u) dx &= - \int_{B_R} \Delta u dx = - \int_{\partial B_R} \frac{\partial u}{\partial \nu} dS \\ &\geq \varepsilon |\partial B_R| = \varepsilon |\partial B_1| R^{r-1}. \end{aligned}$$

Since $r \geq 2$, the right-hand side tends to infinity as $R \rightarrow \infty$, contradicting (2.6). It follows that $r = 1$, and this completes the proof. $\square$

3. PROOF OF THE MAIN THEOREM

In this section, we remove the assumption in Lemma 2.2 that a concave solution attains a local maximum and prove Theorem 1.1. We first recall the notion of stability used in the proof. A solution u of (1.1) is called stable if

$$(3.1) \quad \int_{\mathbb{R}^n} f'(u) \varphi^2 dx \leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. We begin by showing that if a stable solution is independent of some variables, it remains stable as a solution in the remaining variables, using a cutoff argument similar to that of Ambrosio and Cabré [1], and then prove that every one-dimensional solution of (1.1) is unstable.

Suppose that $n \geq 2$ and u is a stable solution independent of x_n , so that $u(x', x_n) = v(x')$ for $x' \in \mathbb{R}^{n-1}$. For $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ and a nonzero $\eta \in C_c^\infty(\mathbb{R})$, take

$$\varphi(x', x_n) = \psi(x')\eta\left(\frac{x_n}{L}\right), \quad L > 0.$$

Substituting into (3.1) and dividing by $L \int_{\mathbb{R}} \eta^2 ds$, we obtain

$$(3.2) \quad \int_{\mathbb{R}^{n-1}} f'(v)\psi^2 dx' \leq \int_{\mathbb{R}^{n-1}} |\nabla \psi|^2 dx' + \frac{\int_{\mathbb{R}} |\eta'|^2 ds}{L^2 \int_{\mathbb{R}} \eta^2 ds} \int_{\mathbb{R}^{n-1}} \psi^2 dx'.$$

Letting $L \rightarrow \infty$ shows that v is stable in $\mathbb{R}^{n-1}$. The same argument can be repeated when u is independent of several variables.

We now consider an entire solution g of $-g'' = f(g)$ on $\mathbb{R}$. Choose $\eta \in C_c^\infty(\mathbb{R})$ with $\eta \equiv 1$ on $[-1, 1]$. For $L \geq 1$, positivity of f' implies

$$(3.3) \quad \int_{\mathbb{R}} f'(g(s))\eta\left(\frac{s}{L}\right)^2 ds \geq \int_{-1}^1 f'(g(s))ds > 0.$$

On the other hand,

$$(3.4) \quad \int_{\mathbb{R}} \left| \frac{d}{ds} \eta\left(\frac{s}{L}\right) \right|^2 ds = \frac{1}{L} \int_{\mathbb{R}} |\eta'(s)|^2 ds.$$

As $L \rightarrow \infty$, the quantity in (3.4) tends to 0, while the positive lower bound in (3.3) is independent of L . Thus g is unstable, and the preceding argument on stability in lower dimensions shows that every entire one-dimensional solution of (1.1) is also unstable.

We now prove our main theorem, using the stability properties established above.

Proof of Theorem 1.1. Suppose, for a contradiction, that the theorem fails, and choose the smallest dimension $n \geq 2$ for which there is an entire concave solution u in $\mathbb{R}^n$ that is not one-dimensional. By Lemma 2.2, u does not attain its maximum, so the set $\{x : u(x) \geq u(0)\}$ is unbounded. Choose x_j in this set with $|x_j| \rightarrow \infty$, and assume that $x_j/|x_j| \rightarrow e$ for some unit vector e as $j \rightarrow \infty$. For fixed $x \in \mathbb{R}^n$ and $s > 0$, concavity implies, whenever $|x_j| > s$,

$$\begin{aligned} u\left(\left(1 - \frac{s}{|x_j|}\right)x + \frac{s}{|x_j|}x_j\right) &\geq \left(1 - \frac{s}{|x_j|}\right)u(x) + \frac{s}{|x_j|}u(x_j) \\ &\geq \left(1 - \frac{s}{|x_j|}\right)u(x) + \frac{s}{|x_j|}u(0). \end{aligned}$$

Letting $j \rightarrow \infty$, we find that $u(x + se) \geq u(x)$. Rotating coordinates so that e is the x_n -direction, we have $u_n \geq 0$. Differentiating equation (1.1) yields

$$(3.5) \quad -\Delta u_n = f'(u)u_n \geq 0 \quad \text{in } \mathbb{R}^n.$$

The strong maximum principle shows that either $u_n \equiv 0$ or $u_n > 0$. In the first case, u is a concave entire solution in $\mathbb{R}^{n-1}$, hence one-dimensional by the choice of n , a contradiction. Therefore $u_n > 0$.

We next show that u is stable. Indeed, multiplying (3.5) by φ^2/u_n , integrating by parts and applying the Cauchy-Schwarz inequality, we obtain for $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(3.6) \quad \begin{aligned} \int_{\mathbb{R}^n} f'(u) \varphi^2 dx &= \int_{\mathbb{R}^n} \left(\frac{2\varphi}{u_n} \nabla u_n \cdot \nabla \varphi - \frac{\varphi^2}{u_n^2} |\nabla u_n|^2 \right) dx \\ &\leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx. \end{aligned}$$

Following the idea of Ambrosio and Cabré [1], we now consider translations of u in the x_n -direction to obtain a stable limit independent of x_n . For $\ell \geq 0$, define

$$u_\ell(x', x_n) = u(x', x_n + \ell).$$

Each u_ℓ is again a stable concave solution of (1.1), and $u_n > 0$ implies that $u_\ell(x)$ increases with ℓ . In particular, for every $x \in \mathbb{R}^n$ and $\ell \geq 0$,

$$(3.7) \quad u_\ell(x) \geq u(x).$$

To obtain a finite limit, it remains to bound u_ℓ above in every ball B_R independently of ℓ . Fix $R > 0$, and choose $\eta \in C_c^\infty(B_{2R})$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_R , and $|\nabla \eta| \leq C/R$. Applying (3.6) to u_ℓ with $\varphi = \eta$ and using $f' \geq cf$, we obtain

$$(3.8) \quad c \int_{B_R} f(u_\ell) dx \leq \int_{B_R} f'(u_\ell) dx \leq C_n R^{n-2},$$

where C_n depends only on n .

For $x, y \in B_R$, the concavity of u_ℓ and (3.7) imply

$$(3.9) \quad u_\ell \left(\frac{x+y}{2} \right) \geq \frac{u_\ell(x) + u_\ell(y)}{2} \geq \frac{u_\ell(x) + \min_{\bar{B}_R} u}{2}.$$

For fixed $x \in B_R$, setting $z = (x+y)/2$ and using $(x+B_R)/2 \subset B_R$, (3.8), (3.9), and the monotonicity of f , we obtain the estimate

$$(3.10) \quad \begin{aligned} \frac{C_n}{c} R^{n-2} &\geq \int_{B_R} f(u_\ell(z)) dz \\ &\geq 2^{-n} \int_{B_R} f \left(u_\ell \left(\frac{x+y}{2} \right) \right) dy \\ &\geq 2^{-n} |B_R| f \left(\frac{u_\ell(x) + \min_{\bar{B}_R} u}{2} \right). \end{aligned}$$

As shown in Section 2, f is a strictly increasing bijection from $\mathbb{R}$ onto $(0, \infty)$ and we denote its inverse by f^{-1} . Applying f^{-1} in (3.10) and using (3.7), we conclude that, for $x \in B_R$,

$$(3.11) \quad \min_{\bar{B}_R} u \leq u_\ell(x) \leq 2f^{-1} \left(\frac{2^n C_n R^{n-2}}{c |B_R|} \right) - \min_{\bar{B}_R} u.$$

Since these bounds are independent of ℓ and $u_\ell(x)$ is increasing in ℓ , it follows that the finite limit

$$(3.12) \quad U(x) = \lim_{\ell \rightarrow \infty} u_\ell(x)$$

exists for every $x \in \mathbb{R}^n$, as $R > 0$ is arbitrary.

We now verify that U solves equation (1.1). Fix $R > 0$ and $p > n$. By (3.11), both u_ℓ and $f(u_\ell)$ are uniformly bounded on B_{4R} . The interior $W^{2,p}$ estimate and Sobolev embedding then give uniform bounds for u_ℓ in $W^{2,p}(B_{3R})$ and $C^{1,\alpha}(B_{2R})$, respectively, for any $0 < \alpha < 1 - n/p$. These standard estimates may be found in [8]. Moreover, $f(u_\ell)$ is uniformly bounded in $C^{0,\alpha}(B_{2R})$. The interior Schauder estimate therefore yields

$$\|u_\ell\|_{C^{2,\alpha}(B_R)} \leq C_R (\|u_\ell\|_{C^0(B_{2R})} + \|f(u_\ell)\|_{C^{0,\alpha}(B_{2R})}),$$

where C_R is independent of ℓ . The uniform $C^{2,\alpha}$ bounds and the Arzelà-Ascoli theorem, together with the pointwise convergence to U in (3.12), imply that $u_\ell \rightarrow U$ in $C^2(\bar{B}_R)$. Since each u_ℓ solves (1.1) and is concave, this convergence shows

$$(3.13) \quad -\Delta U = f(U), \quad \nabla^2 U \leq 0 \quad \text{in } B_R.$$

As $R > 0$ is arbitrary, these conclusions hold throughout $\mathbb{R}^n$. Since $f \in C^\infty(\mathbb{R})$, Schauder estimates yield $U \in C^\infty(\mathbb{R}^n)$.

For fixed $s \in \mathbb{R}$, we have

$$U(x', x_n + s) = \lim_{\ell \rightarrow \infty} u(x', x_n + s + \ell) = U(x', x_n),$$

so U depends only on x' . Moreover, the stability of u_ℓ and its local uniform convergence to U imply

$$\int_{\mathbb{R}^n} f'(U) \varphi^2 dx \leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. The argument leading to (3.2) then shows that U is stable as a solution in $\mathbb{R}^{n-1}$. On the other hand, (3.13) and the minimality of n make U one-dimensional, contradicting the fact established above that no one-dimensional solution of (1.1) is stable. This completes the proof. $\square$

Finally, we prove Corollary 1.2.

Proof of Corollary 1.2. For $f(s) = e^{2s}$, the structural condition (1.3) holds with $c = 2$ and $A = 1$. By Theorem 1.1 and Lemma 2.1, it suffices to solve $-g'' = e^{2g}$ with $g(0) = m$ and $g'(0) = 0$. Set $a = e^m > 0$. Multiplying the equation by $2g'$ and integrating from 0 to s , we obtain

$$(g'(s))^2 + e^{2g(s)} = a^2.$$

Combining this identity with the equation yields $g'' = (g')^2 - a^2$, while $e^{2g} > 0$ implies $|g'| < a$. Integrating $g''/((g')^2 - a^2) = 1$ from 0 to s , with $g'(0) = 0$, we find

$$s = \int_0^{g'(s)} \frac{1}{t^2 - a^2} dt = \frac{1}{2a} \log \frac{a - g'(s)}{a + g'(s)}.$$

Hence $g'(s) = a - 2ae^{2as}/(1 + e^{2as})$. A further integration, together with $g(0) = m = \log a$, yields

$$g(s) = \log a - \log \cosh(as).$$

Substituting into (2.4) proves (1.4), which completes the proof. $\square$

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